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Abstract: In this report we consider a class of the integral matrices H_p and a class of the spectral matrix characteristics $æ_p$, $p \geq 0$. The report contains the proofs of their basic properties. Using these properties, we formulate a criterion for a matrix spectrum to belong to the closed half-plane $\{Re\lambda \leq 0\}$ and prove theorems about qualitative properties of solutions of systems of ordinary differential equations. In particular, we establish new criteria of the asymptotic stability and stability in the sense of Lyapunov for systems of linear ordinary differential equations. We discuss an algorithm to compute the characteristics $æ_p$. We include some examples of systems of linear ordinary differential equations depending on parameters. For these systems, using $æ_p$, we determine the asymptotic stability zones by means of a computer.

Key-words: spectral characteristics of matrices, matrix exponential, stability in the sense of Lyapunov, Lyapunov equation

(Résumé : tsvp)

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A propos de propriétés d'une classe de caractéristiques spectrales des matrices et d'applications aux équations différentielles ordinaires

Résumé : Dans ce rapport on considére une classe de matrices intégrales H_p et une classe des caractéristiques spectrales matricielles $æ_p$, $p \geq 0$. Le rapport contient les preuves de leurs propriétés fondamentalles. En utilisant ces propriétés, nous formulons un critère d'appartenance des spectres matriciels au demi-plan fermé $\{Re\,\lambda \leq 0\}$ et prouvons des théorèmes sur les propriétés qualitatives des solutions des systèmes d'équations différentielles ordinaires. En particulier, on etablit des nouveaux critères de stabilité asymptotique et de stabilité au sens de Lyapounov des solutions des systèmes d'équations différentielles ordinaires linéaires. On présente un algorithm de calcul des paramétres $æ_p$. On inclut quelques exemples de systèmes d'équations différentielles ordinaires linéaires dépendant de paramétres. Pour ces systèmes, utilisant les caractéristiques $æ_p$, on détermine les zones de stabilité au moyen d'un ordinateur.

Mots-clé : caractéristiques spectrales des matrices, exponentielle de matrice, stabilité au sens de Lyapounov, équation de Lyapounov.

1 Introduction

It is well-known that the problem of calculation of eigenvalues of unsymmetric matrices on a computer is "incorrect" (see, e.g., [7, 14]). Therefore, in order to obtain information on the location of the spectrum of a matrix A of order N in the complex plane or on the behaviour of the solutions of the system of ordinary differential equations

$$\frac{dy}{dt} = Ay, \tag{1}$$

we would like to have numerical characteristics for the matrix A which allow us to answer on these questions. One of characteristics of such type is the parameter of the asymptotic stability æ(A) for the system (1). This parameter was introduced by S.K.Godunov and A.Ya.Bulgakov [1, 9]. For a Hurwitz matrix A

$$æ(A) = 2\|A\|\|H\| \tag{2}$$

where the matrix H is the solution of the Lyapunov equation

$$HA + A^*H = -I, \tag{3}$$

I is the unit matrix. If A is non-Hurwitz, then æ$(A) = \infty$. The parameter æ(A) gives a numerical criterion of the asymptotic stability of the solutions of (1). This fact follows from the Lyapunov theorem about the asymptotic stability (see, e.g., [8]).

S.K.Godunov and A.Ya.Bulgakov elaborated [2] an algorithm for calculation of æ(A) on a computer with a guaranteed accuracy. It permits to solve with a guaranteed accuracy the problem of the asymptotic stability for systems of the form (1). This is equivalent to the problem of characterizing the matrices whose spectra belong to the open left half-plane $\{Re\,\lambda < 0\}$. However, the question of characterizing the matrices whose spectra are contained in the closed left half-plane $\{Re\,\lambda \le 0\}$ was open. And the problem of obtaining a numerical criterion of stability in the sense of Lyapunov for the system (1) was not solved too.

Last year one of the authors of the present paper proposed [4] a solution to these problems. An approach for solving is connected with his investigations in the theory of partial differential equations (see, e.g., [3]) and is based upon use of the characteristics $æ_p(A)$, p is real, $p \geq 0$, which he introduced in his lectures on ordinary differential equations in Novosibirsk State University (Russia) in 1987. According to [4] the characteristics $æ_p(A)$ for the matrix A are defined as follows: if the integral

$$H_p = \int_0^\infty (1+t\|A\|)^{-2p} e^{tA^*} e^{tA} dt, \tag{4}$$

exists, then

$$æ_p(A) = a_p \|A\| \|H_p\|, \tag{5}$$

where

$$a_p = \left(\int_0^\infty (1+s)^{-2p} e^{-2s} ds \right)^{-1}.$$

If the integral (4) diverges, then $æ_p(A) = \infty$.

It should be noted here that for the Hurwitz matrices the integral (4) exists for $p = 0$ and is the unique solution of the Lyapunov equation (3), i.e. the parameter $æ_0(A)$ coincides with $æ(A)$.

The spectral characteristics $æ_p(A)$ allow to introduce new criteria [4-6] of the asymptotic stability and stability in the sense of Lyapunov for the solutions of the system (1). Moreover, using the characteristics $æ_p(A)$ for $0 < p \leq 1/2$ instead of $æ(A)$ one can obtain stronger numerical results. Note that the approach from [4] permits also to solve some problems of linear algebra. Thus, the characteristics $æ_p(A)$ permit to answer on the question: Does a matrix spectrum belong to a line (an angle, a strip or a convex polygon) in the complex plane ?

The organization of the paper is as follows. In Sections 2-4 we present the basic results of [4-6]. In particular, we formulate some properties of the matrix H_p in Section 2 and some properties of the characteristics $æ_p(A)$ in Section 3. Section 4 contains some spectral criteria for matrices and the theorems about the properties of the solutions of (1). In Section 5 we discuss a numerical algorithm which allows to obtain two-sided estimates for $æ_p(A)$ by means of a computer. Section 6 presents some numerical examples.

2 Properties of H_p

Let us establish some properties of the matrices H_p. We assume that $A \neq 0$.

Theorem 2.1 *The matrix H_p is Hermitian positive-definite and*

$$\langle H_p v, v\rangle \geq (a_p\|A\|)^{-1}\|v\|^2. \tag{6}$$

Proof. From the definition (4) it follows immediately that the matrix H_p is Hermitian. To prove that the matrix H_p is positive definite we consider the quadratic form $\langle H_p v, v\rangle$ for any vector v. It is not difficult to show that the following identity holds

$$\langle H_p v, v\rangle = \int_0^\infty (1+s\|A\|)^{-2p}\|e^{sA}v\|^2 ds. \tag{7}$$

Indeed, according to (4)

$$\langle H_p v, v\rangle = \langle \left(\int_0^\infty (1+s\|A\|)^{-2p} e^{sA^*}e^{sA} ds\right) v, v\rangle$$

$$= \int_0^\infty (1+s\|A\|)^{-2p}\langle e^{sA^*}e^{sA}v, v\rangle ds = \int_0^\infty (1+s\|A\|)^{-2p}\|e^{sA}v\|^2 ds.$$

Taking into account

$$e^{-|s|\|A\|}\|v\| \leq \|e^{sA}v\|,$$

by (7) we obtain

$$\langle H_p v, v\rangle \geq \int_0^\infty (1+s\|A\|)^{-2p} e^{-2s\|A\|} ds \|v\|^2.$$

This leads to (6). □

Theorem 2.2 *If there exists the integral H_p, then for any $q > p$ the integral H_q exists and*

$$\|H_p\| > \|H_q\|.$$

Proof. Let v_0, $\|v_0\| = 1$, be a vector such that

$$\|H_q\| = \langle H_q v_0, v_0\rangle.$$

From (7) we have

$$\|H_q\| = \int_0^\infty (1+s\|A\|)^{-2q}\|e^{sA}v_0\|^2 ds$$

$$< \int_0^\infty (1+s\|A\|)^{-2p}\|e^{sA}v_0\|^2 ds = \langle H_p v_0, v_0\rangle$$
$$\leq \max_{\|v\|=1} \langle H_p v, v\rangle = \|H_p\|.$$

□

Theorem 2.3 *If for an integer $p \geq 0$ the matrix H_p exists then all eigenvalues of the matrix A have nonpositive real parts, moreover, the maximal size of the Jordan blocks corresponding to the imaginary eigenvalues (if they exist) is not greater than p.*

Proof. Suppose that there exists an eigenvalue λ_j of the matrix A with $Re\,\lambda_j > 0$. Then by (7) for a corresponding eigenvector v_j we have

$$\langle H_p v_j, v_j\rangle = \int_0^\infty (1+s\|A\|)^{-2p}\|e^{sA}v_j\|^2 ds$$
$$= \int_0^\infty (1+s\|A\|)^{-2p} e^{2sRe\,\lambda_j} ds \|v_j\|^2. \tag{8}$$

But if $Re\,\lambda_j > 0$ then this integral doesn't exist. Consequently, the matrix H_p is indefinite. Thus, we have a contradiction, i.e. the eigenvalues cannot belong to the right half-plane.

Assume that there exists an imaginary eigenvalue λ_j of the matrix A for which the size of the corresponding Jordan block is equal to q, $q > p$. Then the system (1) has a solution of the form

$$y(t) = e^{t\lambda_j}(t^{q-1}v_1 + \ldots + v_q) = e^{tA}v_q.$$

According to (7) for the generalized eigenvector v_q we have

$$\langle H_p v_q, v_q\rangle = \int_0^\infty (1+s\|A\|)^{-2p}\|e^{sA}v_q\|^2 ds$$
$$= \int_0^\infty (1+s\|A\|)^{-2p}\|s^{q-1}v_1 + \ldots + v_q\|^2 ds.$$

Since v_1 is an eigenvector of the matrix A, this integral diverges for $q > p$. Hence, the matrix H_p is indefinite. We have a contradiction. □

Note that if the spectrum of the matrix A belongs to the left half-plane and if there exists at least one imaginary eigenvalue, then by the Lyapunov theorem about the asymptotic stability the minimal number $p = p_{min}$ such that there exists a matrix of the form (4) must be greater than zero. The following two theorems assert that p_{min} is uniquely defined by the maximal size of the Jordan blocks corresponding to the imaginary eigenvalues.

Theorem 2.4 *If p is the minimal natural number such that there exists a matrix of the form (4), then the matrix A has at least one imaginary eigenvalue. Moreover, the maximal size of the Jordan blocks corresponding to the imaginary eigenvalues is p.*

Theorem 2.5 *Let all eigenvalues of A belong to the closed left half-plane and there exists at least one imaginary eigenvalue. If the maximal size of the Jordan blocks corresponding to the imaginary eigenvalues is p, then H_p exists and H_{p-1} doesn't exist.*

The proofs of Theorems 2.4 and 2.5 will be obtained in Section 3 as simple corollaries of some theorems about properties of the spectral characteristics $æ_p$.

Theorem 2.6 *If the matrix H_p exists, then the following relation holds*

$$H_p A + A^* H_p = -I + 2p\|A\| H_{p+1/2}. \tag{9}$$

Proof. Introduce the matrix

$$H_p(t) = \int_0^t (1 + s\|A\|)^{-2p} e^{sA^*} e^{sA} ds. \tag{10}$$

We now prove that

$$H_p(t)A + A^* H_p(t) = (1 + t\|A\|)^{-2p} e^{tA^*} e^{tA} - I + 2p\|A\| H_{p+1/2}(t). \tag{11}$$

Indeed, using properties of the matrix exponential we have

$$H_p(t)A + A^* H_p(t) = \int_0^t (1 + s\|A\|)^{-2p} \frac{d}{ds} \left(e^{sA^*} e^{sA} \right) ds$$

$$= (1 + t\|A\|)^{-2p} e^{tA^*} e^{tA} - I + 2p\|A\| \int_0^t (1 + s\|A\|)^{-2p-1} e^{sA^*} e^{sA} ds.$$

Hence, by the definition of $H_{p+1/2}(t)$ we obtain (11). From Theorem 2.3

$$\|e^{tA}\| \le c(1+t)^{p-1}, \qquad t \ge 0.$$

Since

$$H_p(t) \to H_p,$$

$$H_{p+1/2}(t) \to H_{p+1/2}$$

with $t \to +\infty$, we have (9).

□

Theorem 2.7 *For the norm of the matrix H_p the following estimate from below holds*

$$\|H_p\| \ge (a_p\|A\|)^{-1}. \tag{12}$$

Furthermore, if there exists at least one imaginary eigenvalue of the matrix A, then

$$\|H_p\| \ge (2p\|A\|)^{-1}. \tag{13}$$

Note that the inequality (13) is more strict than (12) because for $p \ne 0$

$$a_p^{-1} = \int_0^\infty (1+s)^{-2p} e^{-2s} ds < \int_0^\infty (1+s)^{-2p-1} ds = \frac{1}{2p}.$$

Proof. The inequality (12) follows from (6) as

$$\|H_p\| = \max_{\|v\|=1} \langle H_p v, v\rangle.$$

We now prove (13). Let $y(t)$ be a solution of the system (1), then we have

$$\frac{d}{dt}\langle H_p y(t), y(t)\rangle = \langle (H_p A + A^* H_p) y(t), y(t)\rangle. \tag{14}$$

By (9) the equality (14) can be rewritten as

$$\frac{d}{dt}\langle H_p y(t), y(t)\rangle + \langle y(t), y(t)\rangle = 2p\|A\| \langle H_{p+1/2} y(t), y(t)\rangle.$$

Using the estimate
$$\langle H_p y(t), y(t)\rangle \le \|H_p\|\|y(t)\|^2$$
and Theorem 2.2 we obtain
$$\frac{d}{dt}\langle H_p y(t), y(t)\rangle + \|H_p\|^{-1}\langle H_p y(t), y(t)\rangle \le 2p\|A\|\langle H_p y(t), y(t)\rangle.$$
Rewrite this inequality in the form
$$\frac{d}{dt}\Big(\exp(t\|H_p\|^{-1} - t2p\|A\|)\langle H_p y(t), y(t)\rangle\Big) \le 0.$$
Consequently,
$$\langle H_p y(t), y(t)\rangle \le \exp(t2p\|A\| - t\|H_p\|^{-1})\langle H_p y(0), y(0)\rangle.$$
If there exists at least one imaginary eigenvalue of the matrix A, then for an arbitrary solution $y(t)$ the quadratic form $\langle H_p y(t), y(t)\rangle$ cannot decrease when $t \to +\infty$. Hence, it is necessary that $2p\|A\| - \|H_p\|^{-1} \ge 0$. □

Theorem 2.8 *Let H_q exist. Then for any $p > q + 1/2$ the following estimate holds*
$$\|H_p\| \le ((2p-1)\|A\|)^{-1}(2\|A\|\|H_q\| + 1).$$

Proof. As $p - 1/2 > q$, using Theorem 2.6 we have
$$H_{p-1/2}A + A^* H_{p-1/2} = -I + (2p-1)\|A\|H_p.$$
By Theorem 2.2 we obtain
$$\begin{aligned}\|H_p\| &= ((2p-1)\|A\|)^{-1}\|H_{p-1/2}A + A^*H_{p-1/2} + I\| \\ &\le ((2p-1)\|A\|)^{-1}(\|H_{p-1/2}\|\|A\| + \|A^*\|\|H_{p-1/2}\| + 1) \\ &= ((2p-1)\|A\|)^{-1}(2\|A\|\|H_{p-1/2}\| + 1) \\ &\le ((2p-1)\|A\|)^{-1}(2\|A\|\|H_q\| + 1).\end{aligned}$$
□

According to Theorems 2.2, 2.7 and 2.8 we obtain that if for some p the matrix $H_p = \lim_{t\to+\infty} H_p(t)$ exists, then the family of the norms $\{\|H_{p+q}\|\}$, $q \geq 0$, is strictly decreasing with growth of q. Moreover,

$$\|H_{p+q}\| = O(1/q), \quad q \to \infty.$$

Theorem 2.9 *Let p be the minimal natural number such that there exists the matrix H_p. Then for $t \to +\infty$ we have the following relations*

$$\|H_p - H_p(t)\| = O(t^{-1}), \tag{15}$$

$$\|H_{p+1/2} - H_{p+1/2}(t)\| = O(t^{-2}), \tag{16}$$

$$\|(H_p - H_p(t))A + A^*(H_p - H_p(t))\| = O(t^{-2}) \tag{17}$$

where $H_p(t)$ is defined by (10).

The theorem will be proved in Section 3.

3 Properties of $æ_p$

We now establish some properties of the spectral characteristics $æ_p$ defined by (5). We assume again that $A \neq 0$.

Theorem 3.1 *The following equalities hold*

$$æ_p(A) = a_p\|A\| \sup_{v\neq 0} \left(\int_0^\infty (1 + s\|A\|)^{-2p} \|e^{sA}v\|^2 ds \right) \|v\|^{-2}, \tag{18}$$

$$æ_p(A) = æ_p(A/\|A\|). \tag{19}$$

Proof. The equality (18) follows immediately from the definitions of $æ_p(A)$ and H_p. Let us prove (19).

Since the asymptotic properties of the solutions of the systems

$$\frac{dy}{dt} = Ay \quad \text{and} \quad \frac{dz}{dt} = \frac{A}{\|A\|} z$$

are identical then $æ_p(A/||A||) = \infty$ if and only if $æ_p(A) = \infty$. If $æ_p(A)$ and $æ_p(A/||A||)$ are finite then (19) follows from the definitions of these parameters. Indeed,

$$æ_p(A) = a_p\|A\|\|H_p\| = a_p\|A\|\|\int_0^\infty (1+s\|A\|)^{-2p} e^{sA^*} e^{sA} ds\|$$

$$= a_p\|\int_0^\infty (1+\eta)^{-2p} e^{\eta A^*/\|A\|} e^{\eta A/\|A\|} d\eta\| = æ_p(A/\|A\|).$$

□

Theorem 3.2 *If $æ_p(A) < \infty$, then $1 < æ_q(A) < æ_p(A)$, $q > p$.*

Proof. Using the definition (5) and Theorem 2.2, we have $æ_q(A) < æ_p(A)$, $q > p$. By Theorem 2.1 the matrix H_p is Hermitian positive definite. Therefore

$$\|H_p\| = \max_{\|v\|=1} \langle H_p v, v\rangle.$$

Taking into account (6), we obtain $\|H_p\| \geq (a_p\|A\|)^{-1}$. Hence, $æ_p(A) \geq 1$. Since the family $\{æ_p(A)\}$ is monotonically decreasing with increase of p, it follows that $æ_p(A) > 1$. □

Theorem 3.3 *If the matrix A has at least one eigenvalue λ_j with $Re\,\lambda_j > 0$ then $æ_p(A) = \infty$ for all $p \geq 0$.*

Proof. Let v_j be an eigenvector corresponding to λ_j. By (8) we have

$$\langle H_p v_j, v_j\rangle = \int_0^\infty (1+\|A\|)^{-2p} e^{2sRe\,\lambda_j} ds \|v_j\|^2.$$

Since $Re\,\lambda_j > 0$, it follows that the integral diverges. Consequently, $æ_p(A) = \infty$ for all p. □

Theorem 3.4 *If $æ_N(A) = \infty$, then A has at least one eigenvalue with positive real part.*

Proof. We use an indirect proof. Assume that all eigenvalues have nonpositive real parts, i.e. $Re\,\lambda_j \le 0$, $j = 1, \dots, N$. From the Gelfand-Shilov inequality (see, e.g., [13]) we have

$$\|e^{tA}\| \le \left(1 + \frac{(2t\|A\|)}{1!} + \ldots + \frac{(2t\|A\|)^{N-1}}{(N-1)!}\right), \qquad t \ge 0. \tag{20}$$

That is

$$\langle e^{tA}v, e^{tA}w\rangle \le \left(1 + \frac{(2t\|A\|)}{1!} + \ldots + \frac{(2t\|A\|)^{N-1}}{(N-1)!}\right)^2 \|v\|\|w\|$$

for any vectors v, w. Hence, the quadratic form

$$\langle \left(\int_0^t (1+s\|A\|)^{-2N} e^{sA^*} e^{sA} ds\right) v, w\rangle$$

$$= \int_0^t (1+s\|A\|)^{-2N} \langle e^{sA}v, e^{sA}w\rangle ds$$

has the limit when $t \to +\infty$. Consequently, the matrix H_N exists and $æ_N(A)$ must be finite. Thus, we have a contradiction. □

Theorem 3.5 *Let all eigenvalues of the matrix A have nonpositive real parts. If the maximal size of the Jordan blocks corresponding to the imaginary eigenvalues is p, then $æ_{p-1}(A) = \infty$ and $æ_p(A) < \infty$.*

Proof. Let J be the Jordan canonical form of the matrix A , i.e. $T^{-1}AT = J$. Since the maximal size of the Jordan blocks corresponding to the imaginary eigenvalues is p, then there exists a constant $c > 0$ such that the following estimate holds

$$\|e^{tA}\| = \|Te^{tJ}T^{-1}\| \le \|T\|\|T^{-1}\|\|e^{tJ}\| \le c(1+t\|A\|)^{p-1}, \qquad t \ge 0.$$

Using this inequality, we obtain

$$\langle e^{tA}v, e^{tA}w\rangle \le c^2(1+t\|A\|)^{2p-2}\|v\|\|w\|$$

for any vectors v, w. Since for $H_p(t)$ from (10)

$$\langle H_p(t)v, w\rangle = \int_0^t (1+s\|A\|)^{-2p} \langle e^{sA}v, e^{sA}w\rangle ds,$$

it follows that there exists the limit

$$\lim_{t\to+\infty} \langle H_p(t)v, w\rangle.$$

Consequently, the matrix H_p is definite and $æ_p(A) < \infty$.

Now we prove that $æ_{p-1}(A) = \infty$. Assume that $æ_{p-1}(A) < \infty$. From (18) we have

$$æ_{p-1}(A) = a_{p-1}\|A\| \sup_{v\neq 0} \left(\int_0^\infty (1+s\|A\|)^{-2p+2}\|e^{sA}v\|^2 ds\right) \|v\|^{-2}$$

$$= a_{p-1}\|A\| \sup_{y\neq 0} \left(\int_0^\infty (1+s\|A\|)^{-2p+2}\|Te^{sJ}y\|^2 ds\right) \|Ty\|^{-2}.$$

Hence, for any vector $y \neq 0$,

$$a_{p-1}\|A\| \left(\int_0^\infty (1+s\|A\|)^{-2p+2}\|Te^{sJ}y\|^2 ds\right) \|Ty\|^{-2} \leq æ_{p-1}(A) < \infty.$$

Since the maximal size of the Jordan blocks corresponding to the imaginary eigenvalues is p, then there exists a vector y^0 such that

$$Te^{tJ}y^0 = e^{t\lambda_j}(t^{p-1}v_1^j + \ldots + v_p^j), \qquad Re\,\lambda_j = 0.$$

Consequently, the integral

$$\int_0^\infty (1+s\|A\|)^{-2p+2}\|Te^{sJ}y^0\|^2 ds$$

diverges. Thus, we obtain a contradiction. □

The proof of Theorem 2.5 can then be obtained as a consequence of this Theorem.

Theorem 3.6 *If $æ_{p-1}(A) = \infty$ and $æ_p(A) < \infty$ for natural p, then A has imaginary eigenvalues. Moreover, the maximal size of the Jordan blocks corresponding to the imaginary eigenvalues is p.*

Proof. It is known that all eigenvalues of the matrix A have negative real parts if and only if $æ_0(A) < \infty$. On the other hand, according to Theorem 3.3, the matrix A has not any eigenvalues with positive real parts. Therefore, the spectrum of the matrix A is the union of two sets: the first one consists of all eigenvalues with negative real parts (it may be empty), the second one consists of all imaginary eigenvalues (A can have only imaginary eigenvalues).

Let the maximal size of the Jordan blocks corresponding to the imaginary eigenvalues be $p+k$. From Theorem 3.5 we have $æ_{p+k-1}(A) = \infty$ and $æ_{p+k}(A) < \infty$. Since $æ_p(A) < \infty$, it follows that $k \neq 1$. Taking into account Theorem 3.2, we obtain that k cannot be positive. On the other hand, if $k < 0$, then k cannot be equal to -1 as $æ_{p-1}(A) = \infty$. If $k \leq -2$, then by Theorem 3.2 we have $æ_{p-1}(A) < æ_{p+k}(A) < \infty$. This leads to a contradiction. Hence, k can be only zero. □

The proof of Theorem 2.4 follows immediately from this Theorem.

Theorem 3.7 *If* $æ_{p-1}(A) = \infty$ *and* $æ_p(A) < \infty$ *for natural* p, *then* $æ_{p-1+\varepsilon}(A) < \infty$ *for any* $\varepsilon > 1/2$.

Proof. From Theorem 3.1

$$æ_{p-1+\varepsilon}(A) = a_{p-1+\varepsilon}\|A\| \sup_{\|v\|=1} \int_0^\infty (1+s\|A\|)^{-2p+2-2\varepsilon}\|e^{sA}v\|^2 ds \|v\|^{-2}.$$

Using Theorem 3.6, we have

$$\|e^{sA}v\| \leq c(1+s\|A\|)^{p-1}\|v\|, \qquad s \geq 0. \tag{21}$$

This leads to $æ_{p-1+\varepsilon}(A) < \infty$ for any $\varepsilon > 1/2$. □

Theorem 3.8 *Let p be a nonnegative integer. Then $æ_p(A) < \infty$ if and only if $æ_{p+1/2}(A) < \infty$.*

Proof. Taking into account Theorem 3.2, it is sufficient to verify that if $æ_{p+1/2}(A) < \infty$, then $æ_p(A) < \infty$.

First we consider the case of $p = 0$. Let $æ_{1/2}(A) < \infty$. Suppose that $æ_0(A) = \infty$. From Theorem 3.2 we have $æ_1(A) < \infty$. Consequently, by Theorem 3.6 the matrix A has at least one imaginary eigenvalue λ_j. Let v_j, $\|v_j\| = 1$ be a corresponding eigenvector. Using Theorem 3.1, we obtain the inequality

$$a_{1/2}\|A\| \int_0^\infty (1 + s\|A\|)^{-1} ds$$

$$= a_{1/2}\|A\| \int_0^\infty (1 + s\|A\|)^{-1} \|e^{sA} v_j\|^2 ds \|v_j\|^{-2} \le æ_{1/2}(A) < \infty.$$

But the integral is divergent. We have a contradiction.

Consider the case when p is natural. We assume that $æ_p(A) = \infty$. As noted earlier, if $æ_{p+1/2}(A) < \infty$, then $æ_{p+1}(A) < \infty$. Therefore, by Theorem 3.6 we obtain that there exists λ_j with $Re\,\lambda_j = 0$ and linearly independent vectors $v_1, v_2, \dots, v_{p+1}$ such that the vector function

$$y(t) = e^{t\lambda_j}(t^p v_1 + t^{p-1} v_2 + \ldots + v_{p+1})$$

is a solution of the system (1). Using Theorem 3.1, we have the estimate

$$a_{p+1/2}\|A\| \int_0^\infty (1 + s\|A\|)^{-2p-1} \|s^p v_1 + s^{p-1} v_2 + \ldots + v_{p+1}\|^2 ds \|v_{p+1}\|^{-2}$$

$$\le æ_{p+1/2}(A).$$

It can be rewritten as

$$\left(\int_0^\infty (1 + s\|A\|)^{-2p-1} \|s^p v_1 + s^{p-1} v_2 + \ldots + v_{p+1}\|^2 ds \right)^{1/2}$$

$$\le \left(\frac{æ_{p+1/2}(A)}{a_{p+1/2}\|A\|} \right)^{1/2} \|v_{p+1}\|.$$

Taking into account the Minkovskii inequality, we obtain

$$\left| \left(\int_0^\infty (1 + s\|A\|)^{-2p-1} |s|^{2p} \|v_1\|^2 ds \right)^{1/2} \right.$$

$$\left. - \left(\int_0^\infty (1 + s\|A\|)^{-2p-1} \|\sum_{j=0}^{p-1} s^j v_{p-j+1}\|^2 ds \right)^{1/2} \right|$$

$$\leq \left(\frac{æ_{p+1/2}(A)}{a_{p+1/2}\|A\|} \right)^{1/2} \|v_{p+1}\|.$$

Hence,

$$\left(\int_0^\infty (1 + s\|A\|)^{-2p-1} |s|^{2p} ds \right)^{1/2} \|v_1\|$$

$$\leq \sum_{j=0}^{p-1} \left(\int_0^\infty (1 + s\|A\|)^{-2p-1} |s|^{2j} ds \right)^{1/2} \|v_{p-j+1}\|$$

$$+ \left(\frac{æ_{p+1/2}(A)}{a_{p+1/2}\|A\|} \right)^{1/2} \|v_{p+1}\|.$$

But all summands in the right-hand side are finite and the integral from the left is divergent. Thus, we have a contradiction. Hence, $æ_p(A) < \infty$. □

Finally, we prove Theorem 2.9.

Proof. From the definitions of the matrices H_p, $H_p(t)$ we obtain that

$$H_p - H_p(t) = \int_t^\infty (1 + s\|A\|)^{-2p} e^{sA^*} e^{sA} ds.$$

By the conditions of the Theorem, p is the minimal natural number such that the matrix H_p exists, i.e. $æ_{p-1}(A) = \infty$, $æ_p(A) < \infty$. Then by Theorem 3.6 the inequality (21) holds for any vector v. Hence, estimating the integral

$$\int_t^\infty (1 + s\|A\|)^{-2p} \|e^{sA} v\|^2 ds$$

we obtain (15).

Formulae (16) and (17) can be proved in a similar way. □

4 On spectral properties of matrices and qualitative behaviour of solutions of systems of linear ordinary differential equations

This Section includes some spectral criteria and theorems about qualitative properties of the solutions of the system (1). In particular, it contains new criteria of the asymptotic stability and stability in the sense of Lyapunov of the solutions of (1).

At first, we formulate the spectral criteria for the matrix A.

Theorem 4.1 *The spectrum of A belongs to the open left half-plane $\{Re\,\lambda < 0\}$ if and only if there exists $p \in [0, \frac{1}{2}]$ such that $æ_p(A) < \infty$.*

Proof. If all eigenvalues λ_j of A have negative real parts, then by the Lyapunov theorem $æ_0(A) < \infty$.

We now prove that if $æ_p(A) < \infty$ for some $0 \leq p \leq 1/2$, then the spectrum of A is contained in the open left half-plane. If $æ_{1/2}(A) < \infty$, then by Theorem 3.8 $æ_0(A) < \infty$. Hence, according to the Lyapunov theorem, the spectrum of A belongs to the open left half-plane. If there exists $p \in (0, \frac{1}{2})$ such that $æ_p(A) < \infty$, then from Theorem 3.2 we obtain the above case of $æ_{1/2}(A) < \infty$. $\Box$

Theorem 4.2 *The spectrum of A belongs to the closed left half-plane $\{Re\,\lambda \leq 0\}$ if and only if there exists $p \geq 0$ such that $æ_p(A) < \infty$.*

Proof. If all eigenvalues λ_j of A have $Re\lambda_j \leq 0$, then by Theorem 3.5 $æ_p(A) < \infty$ for natural p which is the maximal size of the Jordan blocks corresponding to the imaginary eigenvalues.

Conversely, if $æ_p(A) < \infty$ for some $p \geq 0$, then from Theorem 3.3 the matrix A cannot have any eigenvalues with positive real parts. $\Box$

We now formulate some criteria [5, 6] of the asymptotic stability and stability in the sense of Lyapunov based upon the properties of the spectral characteristics $æ_p(A)$, $p \geq 0$.

Theorem 4.3 *The null solution of the system (1) is asymptotically stable for $t > 0$ if and only if there exists $p \in [0, \frac{1}{2}]$ such that $æ_p(A) < \infty$.*

Proof. Remind that the null solution of (1) is asymptotically stable for $t > 0$ if and only if all eigenvalues of A are contained in the open left half-plane. Therefore, the proof follows immediately from Theorem 4.1. $\Box$

For $p = 0$ Theorem 4.3 is equivalent to the Lyapunov theorem about the asymptotic stability because $æ_0(A) = æ(A)$. In the case of $p > 0$ the present criterion of the asymptotic stability is new. Note that, according to Theorem 3.2, $æ_p(A) < æ_0(A)$. Therefore, it seems to be necessary to use the characteristics $æ_p(A)$, $0 < p \leq 1/2$, in order to obtain stronger results on a computer. This is confirmed by computational experiments. Thus, Section 6 includes a simple example which shows that there exist matrices A whose spectra belong to the open left half-plane but the ratio $æ_0(A)/æ_{1/2}(A)$ can be very large.

Theorem 4.4 *Let $æ_0(A) = \infty$. The null solution of the system (1) is stable in the sense of Lyapunov for $t > 0$ if and only if there exists $p \in (\frac{1}{2}, \frac{3}{2}]$ such that $æ_p(A) < \infty$.*

Proof. According to the classical spectral criterion of stability of solutions for $t > 0$ (see, e.g., [12]), the null solution of (1) is stable in the sense of Lyapunov for $t > 0$ if and only if all eigenvalues λ_j of A have $Re\,\lambda_j \leq 0$ and only one-dimensional Jordan blocks correspond to the imaginary eigenvalues. Taking into account Theorems 3.5 and 3.6, this is equivalent to $æ_1(A) < \infty$. Using Theorems 3.2, 3.7 and 3.8, we obtain that $æ_1(A) < \infty$ if and only if $æ_p(A) < \infty$ for $1/2 < p \leq 3/2$. □

The following theorem contains an integral representation of the solutions of (1). This representation characterizes asymptotic properties when $t \to +\infty$.

Theorem 4.5 *Let $æ_{p-1}(A) = \infty$ and $æ_p(A) < \infty$ for some natural p. Then for a solution $y(t)$ of the system (1) the following relation holds*

$$\begin{aligned} \|y(t)\|^2 &= (1 + t\|A\|)^{2p}[\langle((H_p(t) - H_p)A + A^*(H_p(t) - H_p))y(0), y(0)\rangle \\ &+ 2p\|A\|\langle(H_{p+1/2} - H_{p+1/2}(t))y(0), y(0)\rangle] \end{aligned} \tag{22}$$

where H_p is defined by (4), $H_p(t)$ by (10).

Proof. Since a solution $y(t)$ of (1) may be written in the form $y(t) = e^{tA}y(0)$, it follows from (11) that

$$\begin{aligned} \langle(H_p(t)A &+ A^*H_p(t))y(0), y(0)\rangle \\ = (1 + t\|A\|)^{-2p}\|y(t)\|^2 - \|y(0)\|^2 &+ 2p\|A\|\langle H_{p+1/2}(t)y(0), y(0)\rangle. \end{aligned} \tag{23}$$

From Theorem 2.6

$$I = 2p\|A\|H_{p+1/2} - (H_pA + A^*H_p),$$

then

$$\|y(0)\|^2 = 2p\|A\|\langle H_{p+1/2}y(0), y(0)\rangle - \langle (H_pA + A^*H_p)y(0), y(0)\rangle.$$

Hence, substituting $\|y(0)\|^2$ in (23) we obtain (22). □

If the conditions of Theorem 4.5 are fulfilled, then from (22) and Theorem 2.9 we obtain that the solution $y(t)$ of (1) satisfies the limit relation

$$\|y(t)\| = O(t^{p-1}), \quad \text{for} \quad t \to +\infty. \tag{24}$$

By Theorems 3.5 and 3.6, p is the maximal size of the Jordan blocks corresponding to the imaginary eigenvalues of A. Therefore, (24) can be obtained from the representation for the matrix exponential

$$e^{tA} = Te^{tJ}T^{-1}$$

where J is the Jordan canonical form of A. It should be noted here that one can verify the conditions of Theorem 4.5 without knowledge of the matrix J. This will follow from a computational algorithm described in Section 5.

Finally, we establish uniform estimates for the solutions of (1) on the half-line $t > 0$. These estimates will be essentially used in Section 5 for the foundation of the algorithm for $æ_p(A)$.

Theorem 4.6 *Let $æ_{p-1}(A) = \infty$ and $æ_p(A) < \infty$ for some natural p. Then for a solution $y(t)$ of the system (1) the following estimate holds*

$$\begin{aligned}\langle H_py(t), y(t)\rangle \;&\leq\; (1 + t\|A\|)^{2(p-1+\varepsilon)}[\exp(-t/\|H_p\|)\langle H_py(0), y(0)\rangle \\ &+\; 2(p+1-\varepsilon)\|A\|\|H_p\|\langle H_{p-1+\varepsilon}y(0), y(0)\rangle],\end{aligned} \tag{25}$$

$$t \geq 0, \qquad \varepsilon > 1/2.$$

Proof. Let us consider the form

$$h(t) = (1 + t\|A\|)^{-2(p-1+\varepsilon)}\langle H_p y(t), y(t)\rangle$$

for an arbitrary solution $y(t)$. Since

$$\frac{d}{dt}\,h(t) = (1 + t\|A\|)^{-2(p-1+\varepsilon)}\frac{d}{dt}\,\langle H_p y(t), y(t)\rangle$$
$$-2(p-1+\varepsilon)\|A\|(1 + t\|A\|)^{-1}h(t)$$

and

$$\frac{d}{dt}\,\langle H_p y(t), y(t)\rangle = \langle (H_p A + A^* H_p) y(t), y(t)\rangle,$$

from Theorem 2.6 we have

$$\frac{d}{dt}\,h(t) + (1 + t\|A\|)^{-2(p-1+\varepsilon)}\|y(t)\|^2$$

$$= 2p\|A\|(1 + t\|A\|)^{-2(p-1+\varepsilon)}\langle H_{p+1/2} y(t), y(t)\rangle$$
$$-2(p-1+\varepsilon)\|A\|(1 + t\|A\|)^{-1}h(t).$$

We denote by $f(t)$ the right-hand side of this equality. Using the inequality

$$\langle H_p y(t), y(t)\rangle \le \|H_p\|\|y(t)\|^2,$$

we obtain

$$\frac{d}{dt}\,h(t) + \|H_p\|^{-1}h(t) \le f(t)$$

or

$$\frac{d}{dt}\,(\exp(t/\|H_p\|)h(t)) \le \exp(t/\|H_p\|)f(t).$$

Hence,

$$h(t) \le \exp(-t/\|H_p\|)h(0) + \int_0^t \exp(-(t-s)/\|H_p\|)f(s)\,ds.$$

We now estimate the function $f(t)$. By the monotony property

$$\langle H_{p+1/2} y(t), y(t)\rangle \le \langle H_p y(t), y(t)\rangle$$

and from the definition of $h(t)$ we have

$$f(t) \leq 2p\|A\|(1-(1+t\|A\|)^{-1})h(t)$$

$$+2(1-\varepsilon)\|A\|(1+t\|A\|)^{-1}h(t) \leq 2p\|A\|h(t)$$

$$+2(1-\varepsilon)\|A\|(1+t\|A\|)^{-1}h(t) \leq 2(p+1-\varepsilon)\|A\|h(t).$$

This leads to the inequality

$$h(t) \leq \exp(-t/\|H_p\|)h(0)$$

$$+2(p+1-\varepsilon)\|A\| \int_0^t \exp(-(t-s)/\|H_p\|)h(s)\, ds$$

$$\leq \exp(-t/\|H_p\|)h(0)$$

$$+2(p+1-\varepsilon)\|A\| \int_0^t (1+s\|A\|)^{-2(p-1+\varepsilon)} \langle H_p y(s), y(s)\rangle ds.$$

As $y(s) = e^{sA}y(0)$, by Theorem 3.7 from the previous section we obtain

$$h(t) \leq \exp(-t/\|H_p\|)h(0) + 2(p+1-\varepsilon)\|A\|\, \|H_p\| \langle H_{p-1+\varepsilon} y(0), y(0)\rangle.$$

Consequently, from the definition $h(t)$ we have (25). □

Corollary 4.1 *The following estimate holds*

$$\|e^{tA}\|^2 \leq (1 \quad + \quad t\|A\|)^{2(p-1+\varepsilon)}(\lambda^p_{max}/\lambda^p_{min})$$
$$\times[\exp(-t/\lambda^p_{max}) \quad + \quad 2(p+1-\varepsilon)\lambda^{(p-1+\varepsilon)}_{max}\|A\|], \tag{26}$$
$$t \geq 0, \qquad 1 \geq \varepsilon > 1/2,$$

where λ^p_{max} and λ^p_{min} are the maximal and the minimal eigenvalues of the matrix H_p respectively, $\lambda^{p-1+\varepsilon}_{max}$ is the maximal eigenvalue of the matrix $H_{p-1+\varepsilon}$.

The estimate (26) is an analog of the Gelfand-Shilov estimate (20).

5 Numerical algorithm for $æ_p(A)$

From the definitions of the numerical characteristics $æ_p(A)$ it follows that $æ_p(A) < \infty$ if and only if the integral (4) converges. Since the matrix

$$(1 + s\|A\|)^{-2p} e^{sA^*} e^{sA} ds$$

is Hermitian positive definite, therefore, in order to prove convergence of the integral (4), it is sufficient to obtain the estimate from above

$$\|H_p(t)\| \leq const, \quad t \geq 0,$$

where $H_p(t)$ is defined by (10). Note that by Theorem 3.1, one can suppose that $\|A\| = 1$.

We further assume that p is a natural number (see the case of $p = 0$ in [2]).

Consider the sequence $\{H_p(m)\}$ where

$$H_p(m) = \int_0^m (1 + s)^{-2p} e^{sA^*} e^{sA} ds, \qquad \|A\| = 1. \tag{27}$$

Then

$$æ_p(A) = a_p \lim_{m \to \infty} \|H_p(m)\|.$$

Rewrite the integral (27) in the form

$$H_p(m) = \sum_{k=1}^{m} \int_{k-1}^{k} (1 + s)^{-2p} e^{sA^*} e^{sA} ds.$$

It is obvious that the matrices

$$\int_{k-1}^{k} e^{sA^*} e^{sA} ds, \qquad \int_{k-1}^{k} (1 + s)^{-2p} e^{sA^*} e^{sA} ds$$

are Hermitian positive definite and the following estimates hold

$$(1 + k)^{-2p} \int_{k-1}^{k} e^{sA^*} e^{sA} ds < \int_{k-1}^{k} (1 + s)^{-2p} e^{sA^*} e^{sA} ds$$

$$< k^{-2p} \int_{k-1}^{k} e^{sA^*} e^{sA} ds, \quad k \geq 1, \quad p > 0.$$

Therefore, for the matrix $H_p(m)$ we have the estimates

$$\sum_{k=1}^{m}(1+k)^{-2p}\int_{k-1}^{k} e^{sA^*}e^{sA}ds \le H_p(m) \le \sum_{k=1}^{m} k^{-2p}\int_{k-1}^{k} e^{sA^*}e^{sA}ds.$$

As

$$0 < k^{-2p} - (1+k)^{-2p} \le k^{-2p}(1-1/4^p),$$

then the convergence

$$\lim_{m\to\infty}\|H_p(m)\|$$

is equivalent to the convergence of the series

$$\sum_{k=1}^{\infty} k^{-2p}\int_{k-1}^{k} e^{sA^*}e^{sA}ds. \tag{28}$$

Consider the following integrals

$$B_k = \int_{k-1}^{k} e^{sA^*}e^{sA}ds, \qquad k \ge 1.$$

Having calculated B_1 and e^A we can determine other integrals B_k, $k \ge 2$, by means of the relations

$$B_k = e^{(k-1)A^*}B_1 e^{(k-1)A}.$$

Hence, we have the simple formula for calculation of the partial sum S_m of the series (28)

$$S_m = \sum_{k=1}^{m} k^{-p}e^{(k-1)A^*}B_1 k^{-p}e^{(k-1)A}. \tag{29}$$

However, for solving real problems, it may be better to estimate the characteristic $æ_{p+1/2}(A)$ instead of $æ_p(A)$. By Theorem 3.8 both these characteristics give the same information about the spectrum of the matrix A. On the other hand, by Theorem 3.2 we have $1 < æ_{p+1/2}(A) < æ_p(A)$. Therefore, it is possible that for certain matrix A we will be able to calculate $æ_{p+1/2}(A)$ and won't be able to calculate $æ_p(A)$ by means of a computer because its value is equal to infinity for given computer. Section 6 contains one example which shows that such situation is quite real even for matrix of order two.

For $\text{æ}_{p+1/2}(A)$ we use the same scheme as for $\text{æ}_p(A)$. In this case it is necessary to consider the integrals

$$H_{p+1/2}(m) = \int_0^m (1+s)^{-2p-1} e^{sA^*} e^{sA} ds, \quad \|A\| = 1,$$

instead of the integrals (27) and

$$\text{æ}_{p+1/2}(A) = a_{p+1/2} \lim_{m\to\infty} \|H_{p+1/2}(m)\|.$$

We obtaine analogously that existence of the limit

$$\lim_{m\to\infty} \|H_{p+1/2}(m)\|$$

is equivalent to the convergence of the series

$$\sum_{k=1}^{\infty} k^{-2p-1} \int_{k-1}^{k} e^{sA^*} e^{sA} ds \tag{30}$$

and for the partial sum S'_m we have

$$S'_m = \sum_{k=1}^{m} k^{-p-1/2} e^{(k-1)A^*} B_1 k^{-p-1/2} e^{(k-1)A}.$$

Note that if $\text{æ}_{p-1}(A) = \infty$ and $\text{æ}_p(A) < \infty$ for natural p then one can point out a majorant convergent series for

$$\sum_{k=1}^{\infty} \|k^{-2p-1} \int_{k-1}^{k} e^{sA^*} e^{sA} ds\|. \tag{31}$$

Indeed, according to the inequality (26), we have the following estimates

$$\|e^{(k-1)A}\|^2 \leq k^{2(p-1+\varepsilon)} \frac{\lambda_{max}^p}{\lambda_{min}^p}$$

$$\times \left(e^{-(k-1)/\lambda_{max}^p} + 2(p+1-\varepsilon)\lambda_{max}^{(p-1+\varepsilon)}\right) \leq c(\varepsilon) k^{2(p-1+\varepsilon)},$$

$$1 > \varepsilon > 1/2.$$

Consequently,

$$\|k^{-2p-1}\int_{k-1}^{k} e^{sA^*}e^{sA}ds\|$$
$$= \|k^{-p-1/2}e^{(k-1)A^*}B_1k^{-p-1/2}e^{(k-1)A}\| \leq c(\varepsilon)\|B_1\|k^{2\varepsilon-3}.$$

Therefore, for any $1 > \varepsilon > 1/2$, the series

$$\sum_{k=1}^{\infty} c(\varepsilon)\|B_1\|k^{2\varepsilon-3}$$

is a majorant convergent one for (31). Hence, if $æ_{p-1}(A) = \infty$ and $æ_p(A) < \infty$, then one can estimate the convergence rate of the series (30).

The described algorithm is not optimal, but it demonstrates a principal possibility of obtaining estimates for the spectral characteristics $æ_p(A)$ by means of a computer.

6 Numerical Examples

In this Section we illustrate the practical efficiency of the spectral characteristics $æ_p(A)$ for study of stability of solutions of systems of ordinary differential equations on a computer.

Example 1. Consider the following matrix

$$A = \begin{pmatrix} -0.001 & b \\ 0 & -0.001 \end{pmatrix}$$

where b is a parameter. It is clear that the eigenvalues of A are equal to -0.001 for any b. It is interesting to note that the values of the characteristics $æ_0(A)$ and $æ_{1/2}(A)$ increase with growth of b. The following table contains some values of $æ_0(A)$ and $æ_{1/2}(A)$ with respect to the parameter b.

b	**order of** $æ_0(A)$	**order of** $æ_{1/2}(A)$
1	10^9	10^6
10	10^{12}	10^8
10^2	10^{15}	10^{10}
10^3	10^{18}	10^{12}
10^4	10^{21}	10^{14}

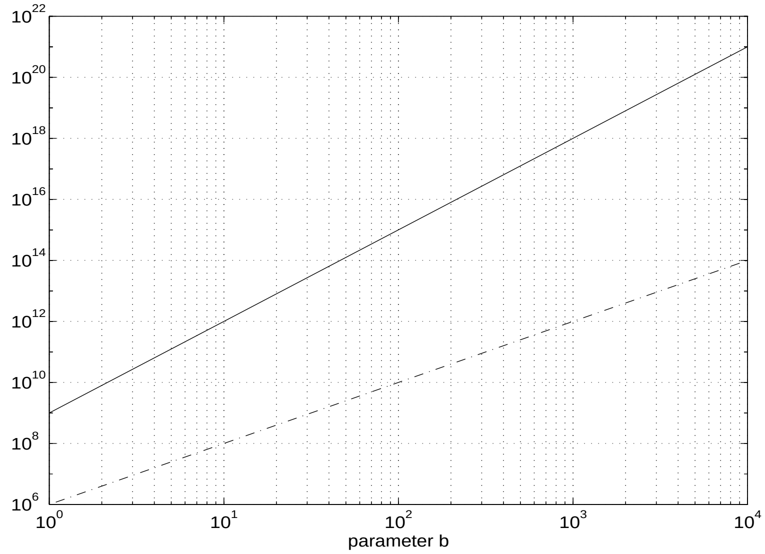

Figure 1: Graphs of the characteristics $æ_0(A)$ and $æ_{1/2}(A)$

One can see that $æ_0(A)$ grows faster than $æ_{1/2}(A)$. Thus, for example,

$$\frac{æ_0(A)}{æ_{1/2}(A)} \sim 10^3 \quad \text{for } b = 1$$

but

$$\frac{æ_0(A)}{æ_{1/2}(A)} \sim 10^7 \quad \text{for } b = 10^4.$$

Hence, the use of the characteristic $æ_{1/2}(A)$ instead of $æ_0(A)$ permits to obtain more rigorous results when solving the problem about the asymptotic stability by means of a computer.

Examples 2 - 6 illustrate a possibility of picking out the asymptotic stability zones for systems of ordinary differential equations depending on two parameters. Figures 2-9 represent the asymptotic stability regions calculated by means of a computer by using the characteristics $æ_0(A)$ and $æ_{1/2}(A)$. Thus, on the figures having the even numbers one can see the asymptotic stability regions obtained by using $æ_0(A)$, on the figures having the odd numbers —

those obtained by using $æ_{1/2}(A)$. We point out also the change character of these characteristics:

point-types	**order of $æ_0(A)$ ($æ_{1/2}(A)$)**
.	10^1
o	10^2
*	10^3
+	10^4
x	10^5

It should be noted that the orders of the parameters $æ_0(A)$ and $æ_{1/2}(A)$ coincide in the interior subregions. However, near boundaries values of $æ_0(A)$ are greater than values of $æ_{1/2}(A)$ by some orders (see the figures on pp.32-41).

In examples 2-5 we consider the system of ordinary differential equations

$$x'' + eRx' + (K + vF)x = 0 \tag{32}$$

where R, K, F are square matrices of order 4, v, e are parameters. In particular, the system of Lagrangian equations is reduced to the system of such type (see, e.g., [11]). The system (32) can be rewritten in the form

$$\frac{dy}{dt} = A(v,e)y$$

where

$$y = \begin{pmatrix} x \\ x' \end{pmatrix}, \quad A(v,e) = \begin{pmatrix} 0 & I \\ -(K+vF) & -eR \end{pmatrix}.$$

Calculating $æ_0(A(v,e))$ and $æ_{1/2}(A(v,e))$ for each fixed pair (v,e) we indicate the asymptotic stability zones in ranges of variation of these parameters.

Example 2.

$$K = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & 3 & 0 \\ 0 & 0 & 0 & 4 \end{pmatrix}, \quad R = \begin{pmatrix} 1 & -0.5 & 0 & 0 \\ -0.5 & 2 & 0 & 0 \\ 0 & 0 & 3 & 0.5 \\ 0 & 0 & 0.5 & 4 \end{pmatrix},$$

$$F = \begin{pmatrix} 1 & 0 & 2 & 0 \\ 0 & 2 & 0 & -2 \\ -2 & 0 & 3 & 0 \\ 0 & 2 & 0 & 4 \end{pmatrix}.$$

Example 3.

$$K = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & 3 & 0 \\ 0 & 0 & 0 & 4 \end{pmatrix}, \quad R = \begin{pmatrix} 1 & -1 & 0 & 2 \\ -1 & 2 & 1 & 0 \\ 0 & 1 & 3 & 0.5 \\ 2 & 0 & 0.5 & 4 \end{pmatrix},$$

$$F = \begin{pmatrix} 1 & 0 & 4 & 0 \\ 0 & 2 & 0 & 0 \\ 4 & 0 & 3 & 0 \\ 0 & 0 & 0 & 4 \end{pmatrix}.$$

Example 4.

$$K = \begin{pmatrix} 0.5 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 0.5 \end{pmatrix}, \quad R = \begin{pmatrix} 1 & -1 & -2 & 0 \\ -1 & 2 & 1 & 0 \\ -2 & -1 & 3 & 0.5 \\ 0 & 0 & -0.5 & 4 \end{pmatrix},$$

$$F = \begin{pmatrix} 1 & 0 & 2 & 0 \\ 0 & 2 & 0 & 0 \\ 2 & 0 & 3 & 1 \\ 0 & 0 & 1 & 4 \end{pmatrix}.$$

Example 5.

$$K = \begin{pmatrix} 10 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0.5 & 0 \\ 0 & 0 & 0 & 0.5 \end{pmatrix}, \quad R = \begin{pmatrix} 2 & -1 & -2 & 1 \\ -1 & 2 & 1 & 0 \\ -2 & -1 & 2 & 0.5 \\ 1 & 0 & -0.5 & 2 \end{pmatrix},$$

$$F = \begin{pmatrix} 5 & 0 & 7 & -1 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 1 & -1 \\ -1 & -5 & -1 & 5 \end{pmatrix}.$$

Examples 6. Consider the system

$$\begin{cases} D\frac{dy}{dt} + (A_1 + (v^2 - e)D)y - A_2 x = 0 \\ eA_2 y - (A_1 + v^2 D)x = 0 \end{cases}$$

where

$$A_1 = \begin{pmatrix} a_1^1 & b_1^1 & 0 & 0 & c_1^1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ b_1^1 & a_1^2 & b_1^2 & 0 & 0 & c_1^2 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & b_1^2 & a_1^3 & b_1^3 & 0 & 0 & c_1^3 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & b_1^3 & a_1^4 & b_1^4 & 0 & 0 & c_1^4 & 0 & 0 & 0 & 0 \\ c_1^1 & 0 & 0 & b_1^4 & a_1^5 & b_1^5 & 0 & 0 & c_1^5 & 0 & 0 & 0 \\ 0 & c_1^2 & 0 & 0 & b_1^5 & a_1^6 & b_1^6 & 0 & 0 & c_1^6 & 0 & 0 \\ 0 & 0 & c_1^3 & 0 & 0 & b_1^6 & a_1^7 & b_1^7 & 0 & 0 & c_1^7 & 0 \\ 0 & 0 & 0 & c_1^4 & 0 & 0 & b_1^7 & a_1^8 & b_1^8 & 0 & 0 & c_1^8 \\ 0 & 0 & 0 & 0 & c_1^5 & 0 & 0 & b_1^8 & a_1^9 & b_1^9 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & c_1^6 & 0 & 0 & b_1^9 & a_1^{10} & b_1^{10} & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & c_1^7 & 0 & 0 & b_1^{10} & a_1^{11} & b_1^{11} \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & c_1^8 & 0 & 0 & b_1^{11} & a_1^{12} \end{pmatrix},$$

$$a_1^1 = 0.67,\ a_1^2 = 1.22,\ a_1^3 = 1.33,\ a_1^4 = 0.55,\ a_1^5 = 1.22,\ a_1^6 = 2.67,$$

$$a_1^7 = 3.11,\ a_1^8 = 1.44,\ a_1^9 = 0.56,\ a_1^{10} = 1.44,\ a_1^{11} = 1.78,\ a_1^{12} = 0.89,$$

$$b_1^1 = -0.33,\ b_1^2 = -0.44,\ b_1^3 = -0.44,\ b_1^4 = 0.,\ b_1^5 = -0.44,\ b_1^6 = -0.89,$$

$$b_1^7 = -0.89,\ b_1^8 = 0,\ b_1^9 = -0.11,\ b_1^{10} = -0.44,\ b_1^{11} = -0.44,$$

$$c_1^1 = 0.67,\ c_1^2 = 1.22,\ c_1^3 = 1.33,\ c_1^4 = 0.55,\ c_1^5 = 1.22,\ c_1^6 = 2.67,$$

$$c_1^7 = 3.11,\ c_1^8 = -0.44,$$

$$A_2 = \begin{pmatrix} a_2^1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & a_2^2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & a_2^3 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & a_2^4 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ c_2^1 & 0 & 0 & 0 & a_2^5 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & c_2^2 & 0 & 0 & 0 & a_2^6 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & c_2^3 & 0 & 0 & 0 & a_2^7 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & c_2^4 & 0 & 0 & 0 & a_2^8 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & c_2^5 & 0 & 0 & 0 & a_2^9 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & c_2^6 & 0 & 0 & 0 & a_2^{10} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & c_2^7 & 0 & 0 & 0 & a_2^{11} & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & c_2^8 & 0 & 0 & 0 & a_2^{12} \end{pmatrix},$$

$$a_2^1=-1,\ a_2^2=-1.33,\ a_2^3=-1.33,\ a_2^4=-0.33,\ a_2^5=-1.33,\ a_2^6=-2.67,$$
$$a_2^7=-2.67,\ a_2^8=-1.33,\ a_2^9=0,\ a_2^{10}=0,\ a_2^{11}=0,\ a_2^{12}=0,$$
$$c_2^k=-a_2^k,\quad k=1,\ldots,8,$$

$$D=\begin{pmatrix}
3&0&0&0&0&0&0&0&0&0&0&0\\
0&4&0&0&0&0&0&0&0&0&0&0\\
0&0&4&0&0&0&0&0&0&0&0&0\\
0&0&0&1&0&0&0&0&0&0&0&0\\
0&0&0&0&4&0&0&0&0&0&0&0\\
0&0&0&0&0&8&0&0&0&0&0&0\\
0&0&0&0&0&0&8&0&0&0&0&0\\
0&0&0&0&0&0&0&4&0&0&0&0\\
0&0&0&0&0&0&0&0&1&0&0&0\\
0&0&0&0&0&0&0&0&0&4&0&0\\
0&0&0&0&0&0&0&0&0&0&4&0\\
0&0&0&0&0&0&0&0&0&0&0&3
\end{pmatrix}.$$

Systems of such type arise in problems of convection of geological structure (see, e.g., [10]).

7 Conclusion

In this paper we presented the approach from [4] for solving the problem about characterizing location of the spectrum of a matrix A in the closed left half-plane. This approach is based upon the use of the family of the spectral characteristics $æ_p(A),\ p\geq 0$. In our opinion, it is necessary to continue both theoretical and applied researches in this direction. At present, the following investigations can be started:

I. Algorithm with a guaranteed accuracy. To solve concrete problems with the help of a computer it is necessary to take into account the structure of machine representation of real numbers. The general peculiarity of all computers is that any number stored in a computer or generated in intermediate computations is indistinguishable from any other one sufficiently close to it. To elaborate an algorithm with a guaranteed accuracy we must take into consideration this effect. Therefore, it is necessary to investigate in more detail the dependence of properties of the characteristics $æ_p(A)$ on A and p. In particular, it is necessary to elaborate a perturbations theory for these parameters, i.e. to study a character of changes of $æ_p(A)$ for small perturbations of the elements of A and p.

II. Efficiency of the algorithm. The algorithm from Section 5 is preliminary. It demonstrates a principal possibility to estimate the spectral characteristics $æ_p$ on a computer. This is confirmed by a series of experiments. However, to solve stability problems for systems depending on parameters it requires much computer time. Now there exists a real possibility for elaboration of an optimal algorithm. In particular, one can create a more optimal algorithm than that from Section 5. One can also investigate analytic properties of $æ_p(A)$ in order to define the boundaries of the stability zones computing values of $æ_p(A)$ on a rare grid in a range of variation of the parameters.

III. Solving some problems of linear algebra. At present, using the approach from [4], the problems about location of a matrix spectrum on a line, a strip, an angle or a convex polygon (both open and closed) can be solved. For each of these problems one can introduce spectral characteristics which are analogous to $æ_p$. Using properties of these characteristics one can propose a justified algorithm for their calculation on a computer.

The investigations were conducted at the Institute of Mathematics, Siberian Branch of the Russian Academy of Sciences, Novosibirsk, Russia and at IRISA, Rennes, France.

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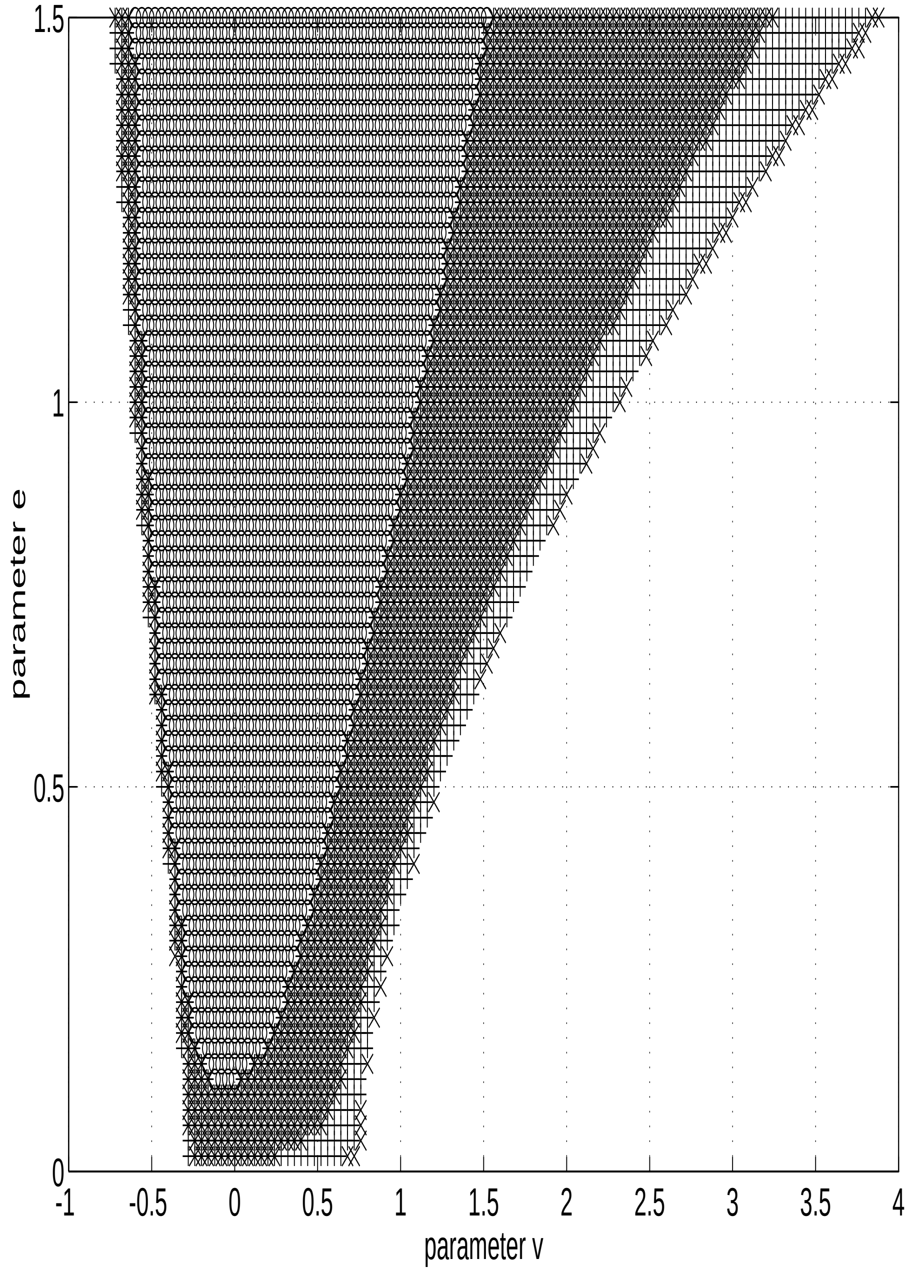

Figure 2: Asymptotic stability zone computated by using $æ_0(A)$

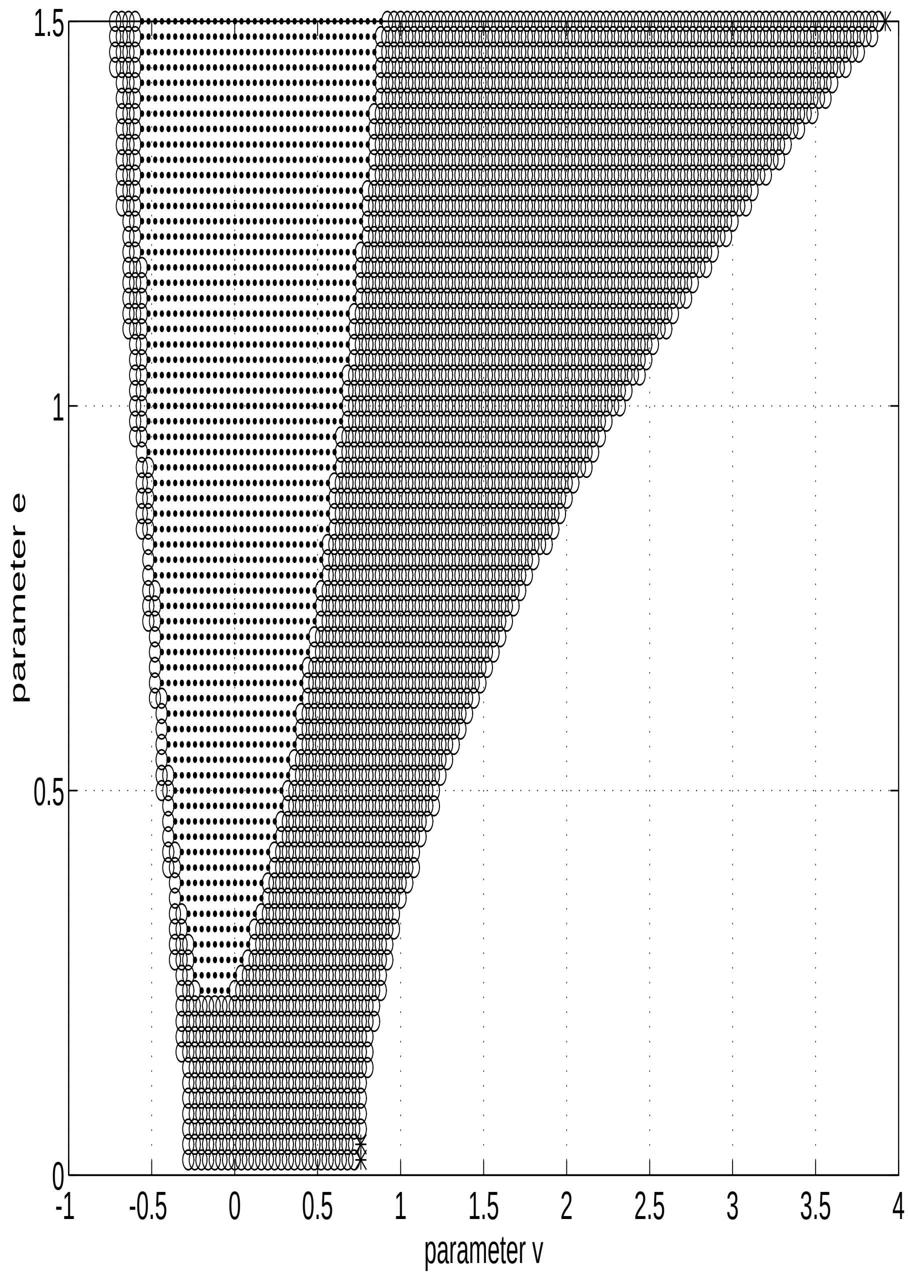

Figure 3: Asymptotic stability zone computated by using $æ_{1/2}(A)$

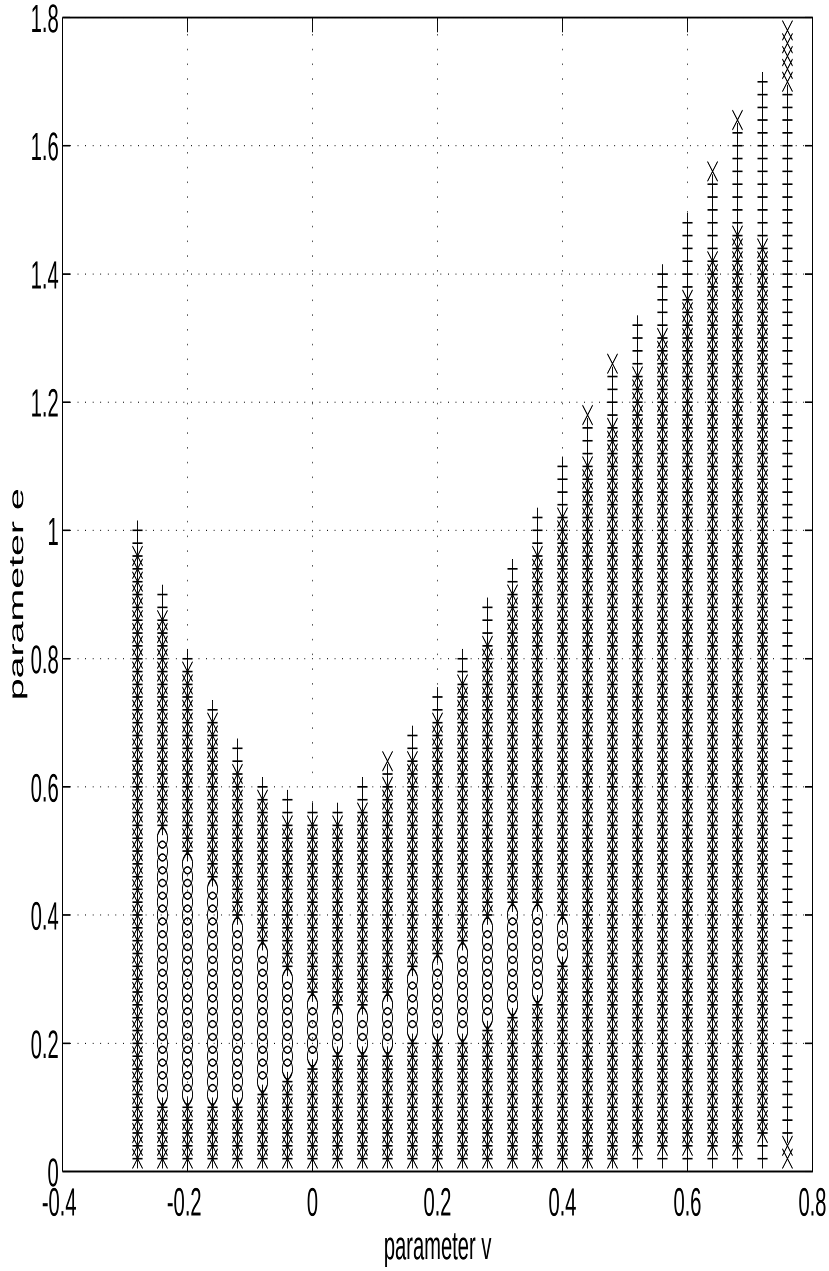

Figure 4: Asymptotic stability zone computated by using $æ_0(A)$

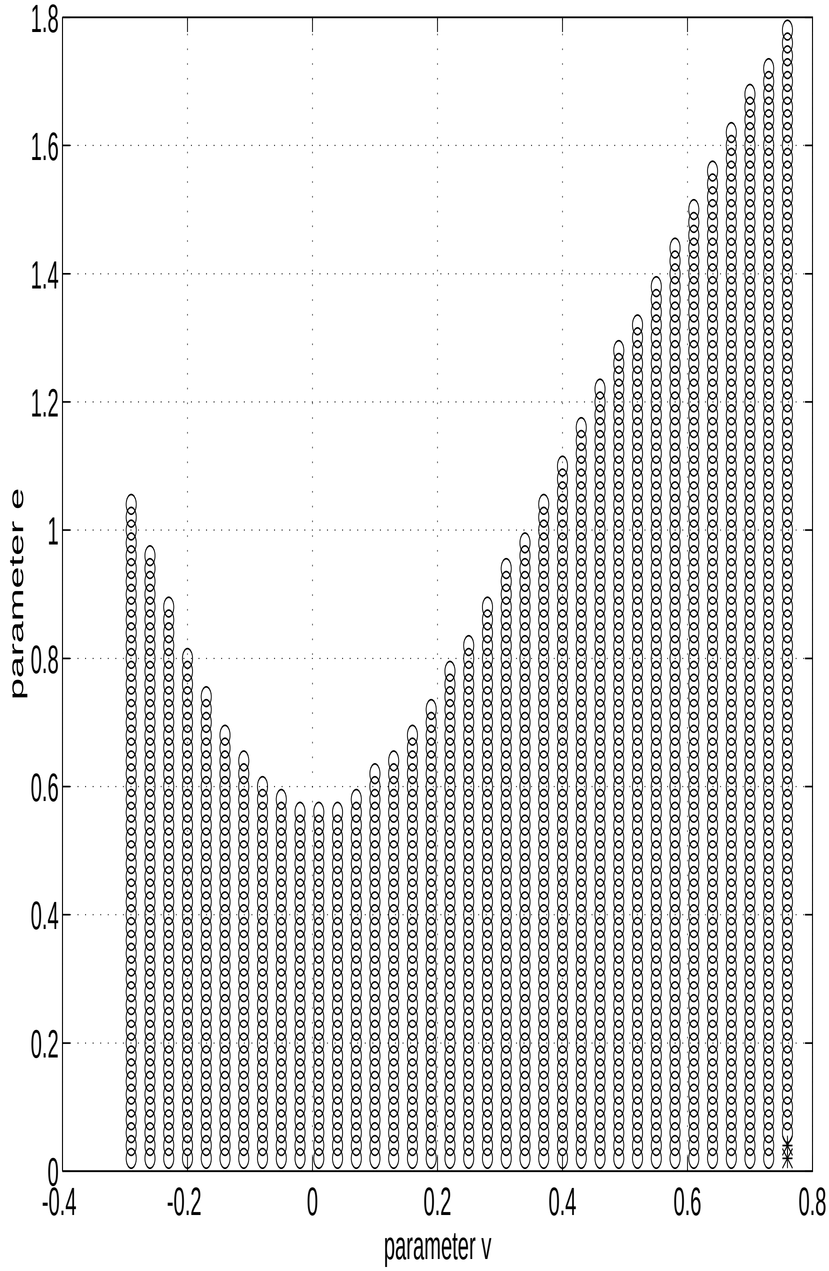

Figure 5: Asymptotic stability zone computated by using $æ_{1/2}(A)$

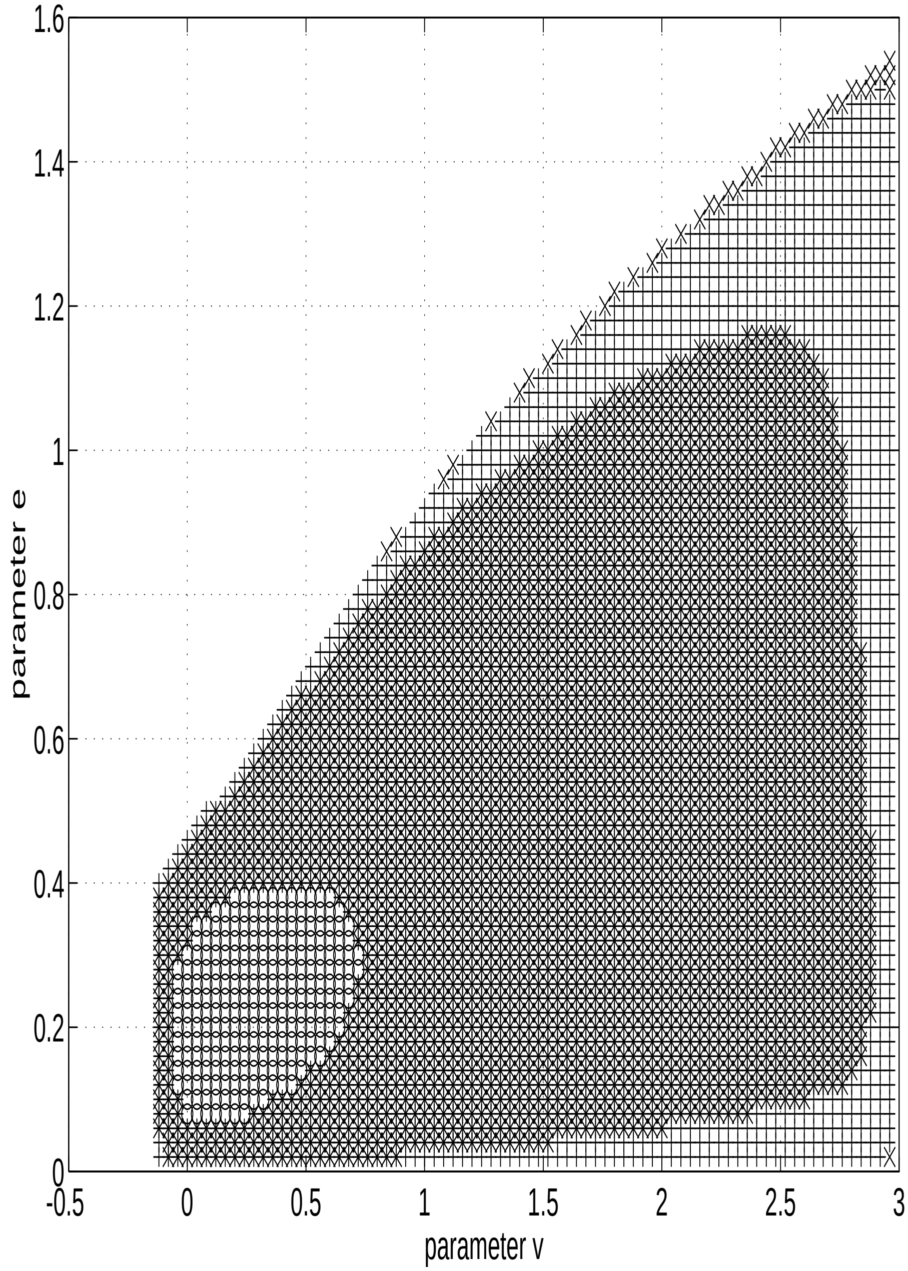

Figure 6: Asymptotic stability zone computated by using $æ_0(A)$

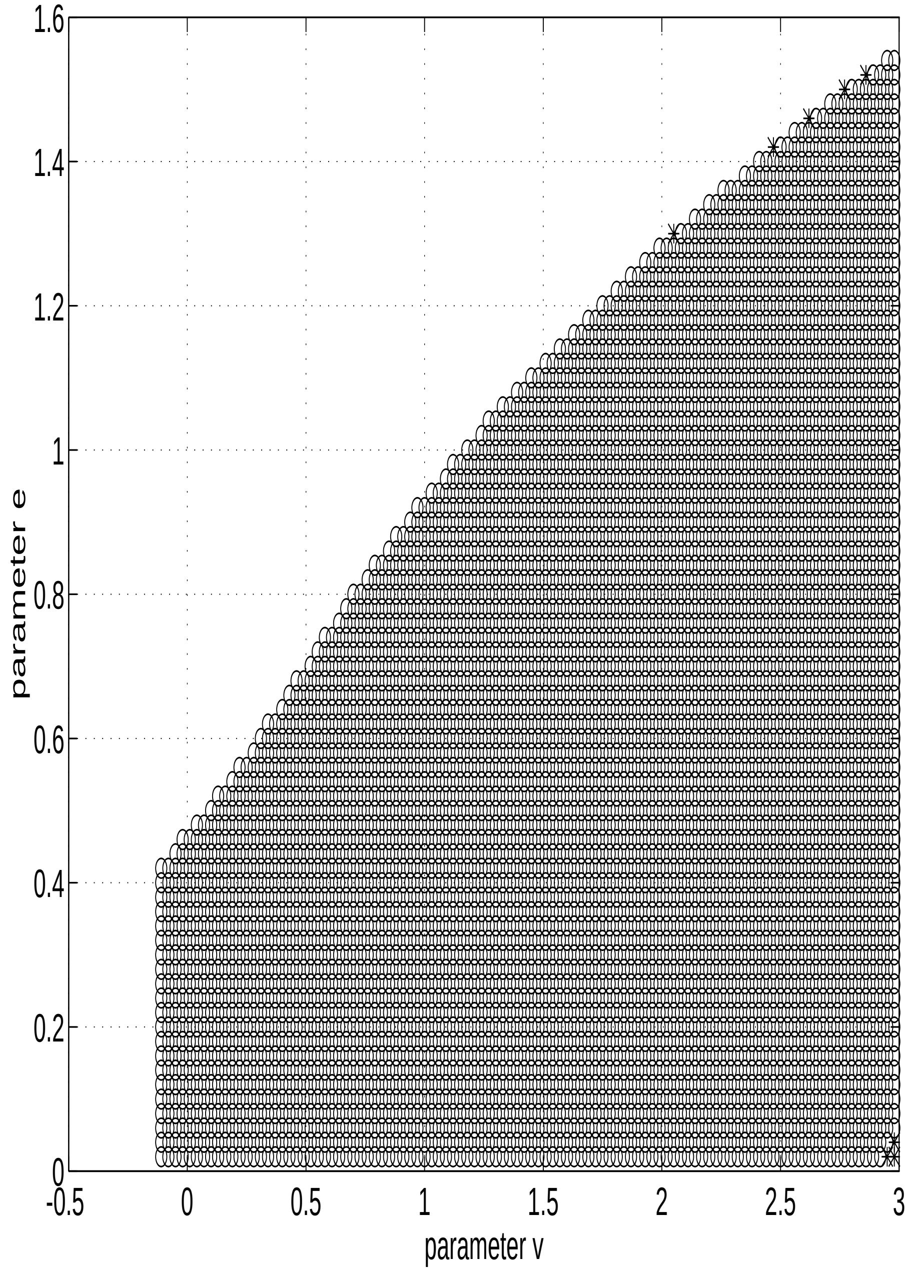

Figure 7: Asymptotic stability zone computated by using $æ_{1/2}(A)$

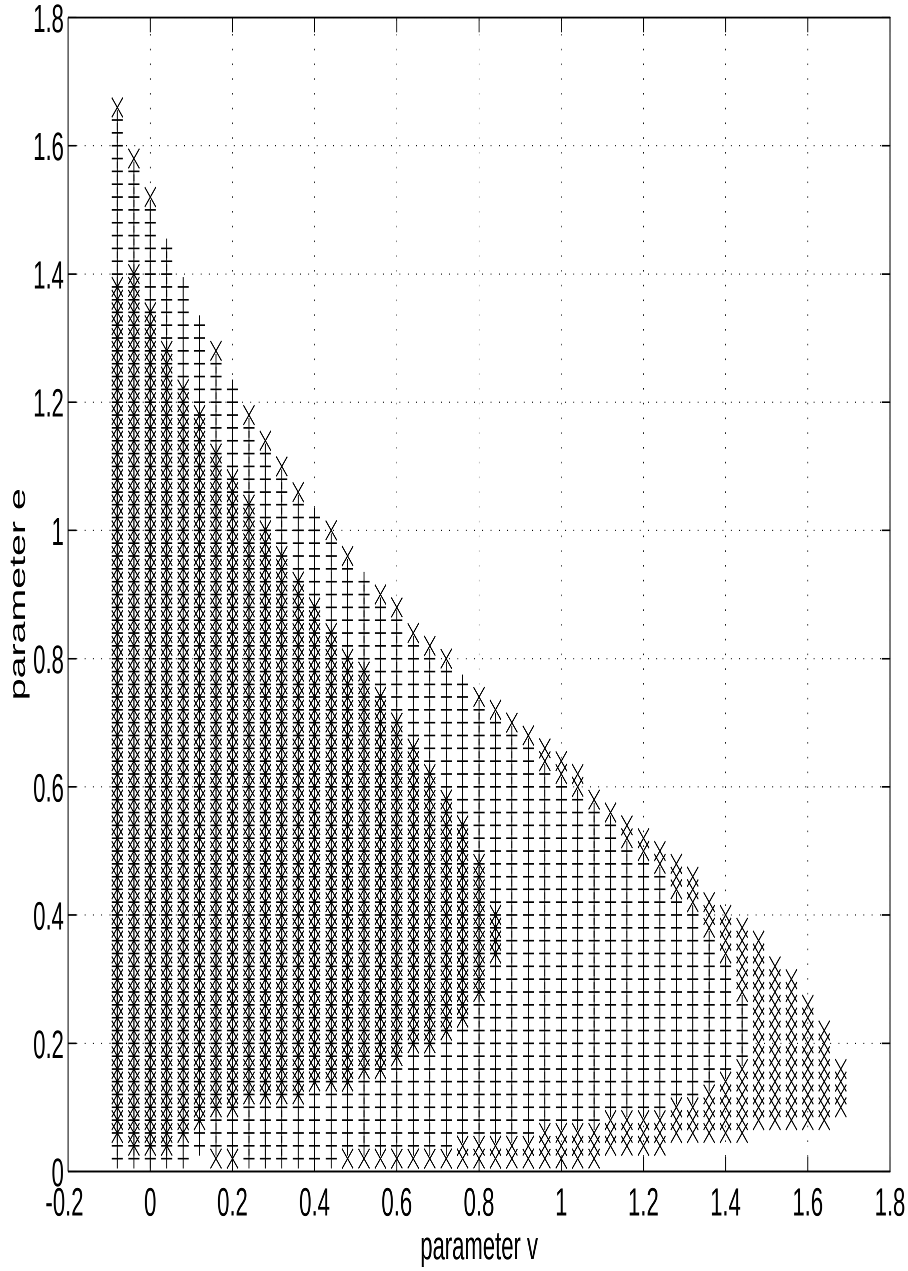

Figure 8: Asymptotic stability zone computated by using $æ_0(A)$

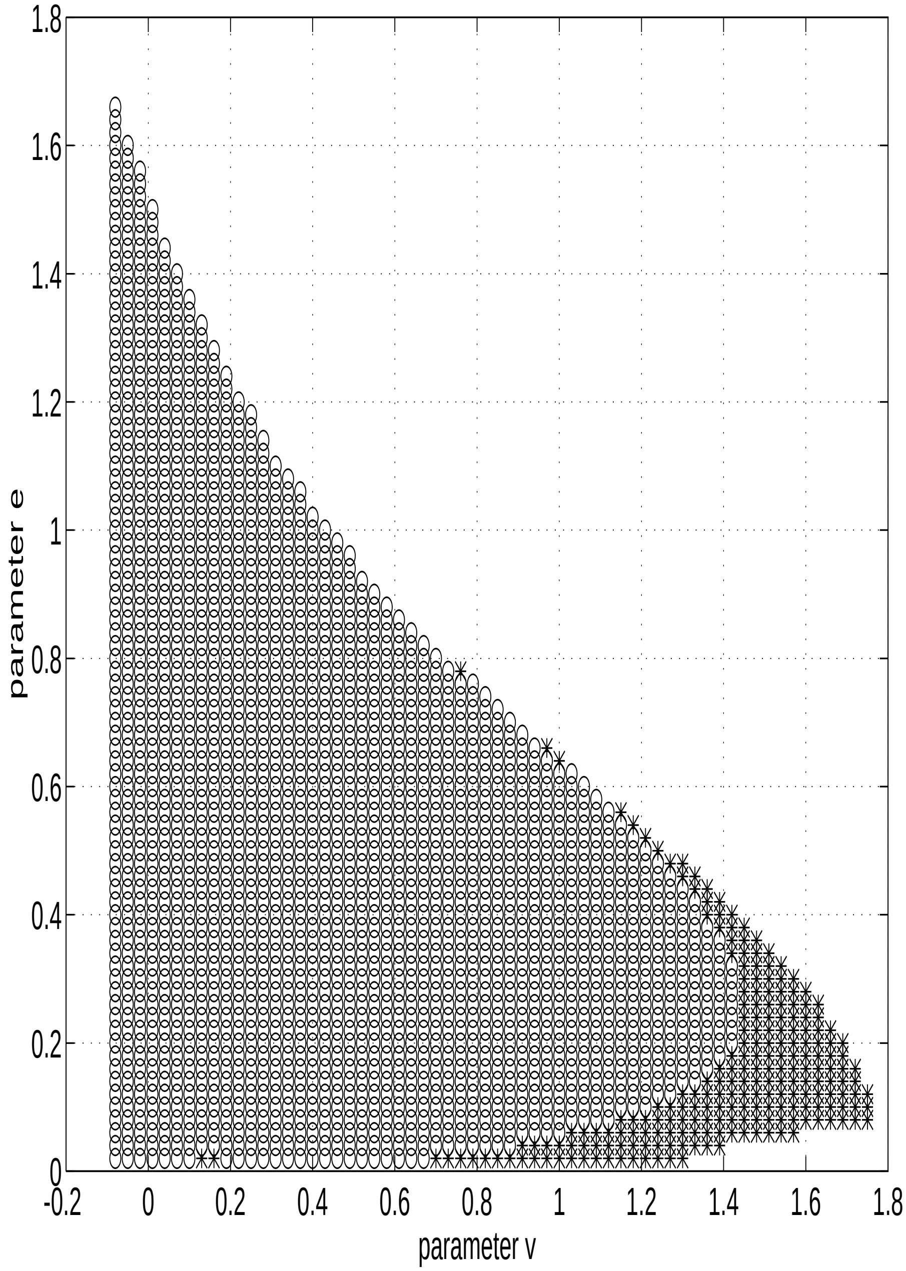

Figure 9: Asymptotic stability zone computated by using $æ_{1/2}(A)$

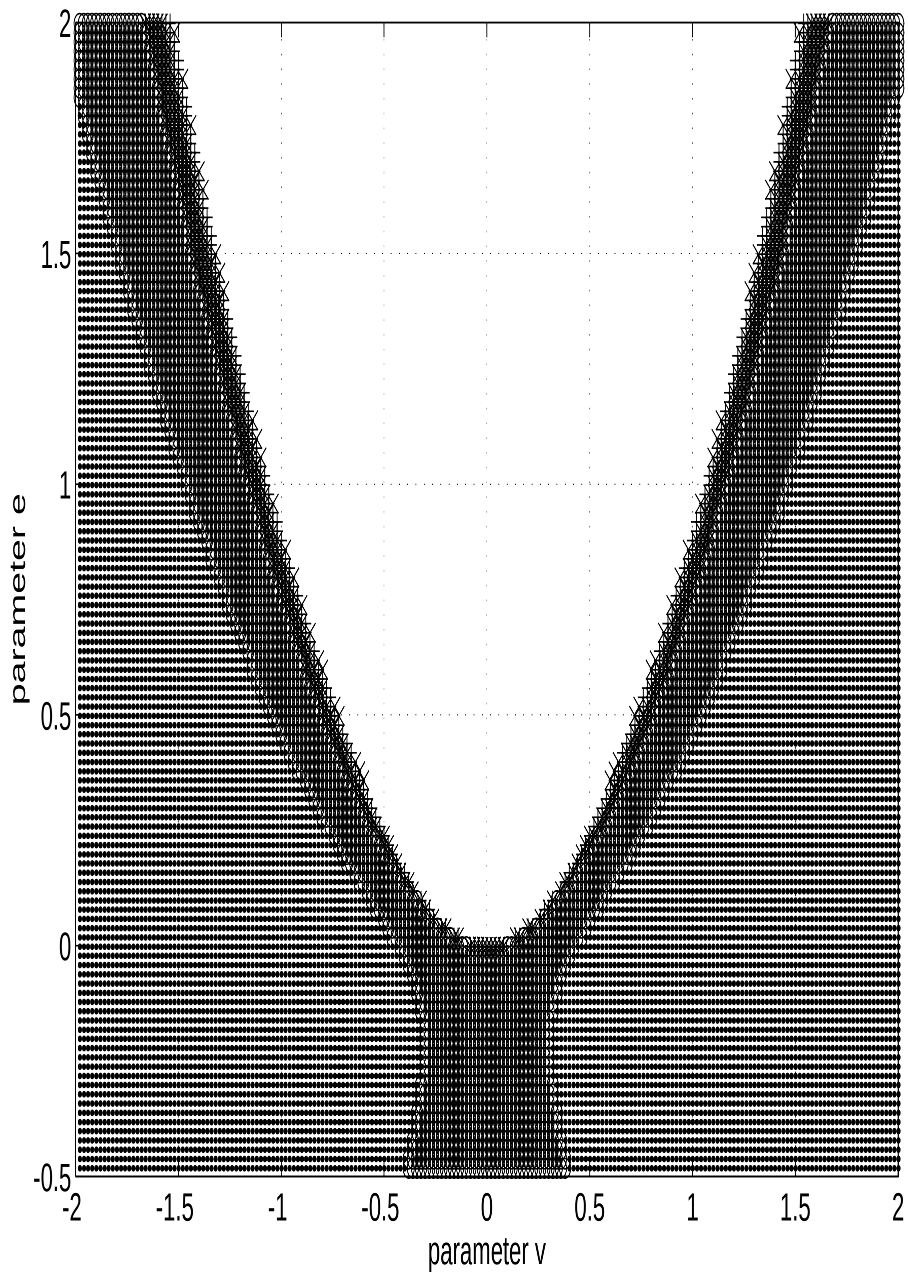

Figure 10: Asymptotic stability zone computated by using $æ_0(A)$

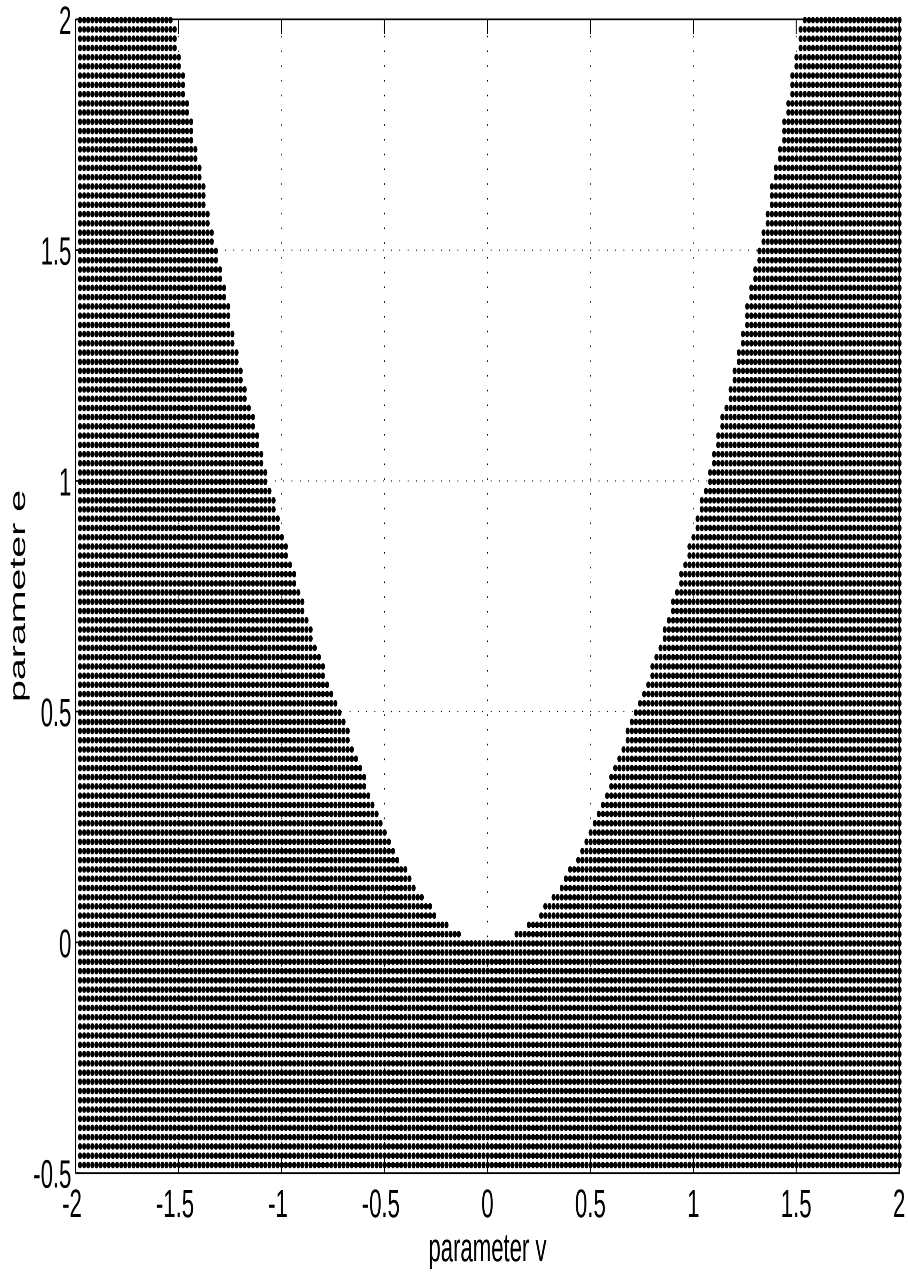

Figure 11: Asymptotic stability zone computated by using $æ_{1/2}(A)$

References

[1] A.Ya. Bulgakov, Effective-calculated parameters of stability quality of linear differential equations system, *Sibirsk. Mat. Zh.*, 21(3): 32-41, 1980 (Russian).

[2] A.Ya. Bulgakov and S.K. Godunov, Calculation of positive definite solutions of the Lyapunov equation, *Trudy Inst. of Math. AN SSSR, Siberian Branch*, 6: 17-38, 1985 (Russian).

[3] G.V. Demidenko, Integral operators defined by boundary value problems for quasielliptic equations, *Dokl. Akad. Nauk* 326(5): 765-769, 1992; English transl. in *Russian Acad. Sci. Dokl. Math.*, 46(2): 343-348, 1993.

[4] G.V. Demidenko, To question about determination of matrices whose spectrums lie on the imaginary axis, Preprint Inst. of Math. Russian Acad. Sci., Siberian Branch, No. 12, 1993 (Russian).

[5] G.V. Demidenko, On a class of spectral characteristics of matrices, *Sibirsk. Mat. Zh.*, (to appear).

[6] G.V. Demidenko, Integrals of Lyapunov type and their applications, *C. R. Acad. Sci.*, (to appear).

[7] D.K. Faddeev and V.N. Faddeeva, *Computational Methods of Linear Algebra*, Fizmatgiz, Moscow, 1963; English transl.: *Computational Methods of Linear Algebra*, W.H.Freeman & Co, San Francisco, 1963.

[8] F.R. Gantmacher , *The Theory of Matrices*, Nauka, Moscow, 1967 (Russian).

[9] S.K. Godunov and A.Ya. Bulgakov, Difficultés de calcul dans le problème de Hurwitz et méthodes pour les surmonter, *Analysis and Optimization of Systems*, Versailles, Springer-Verlag, pp.843-851, 1982.

[10] P. Menegazzi, *Convection naturelle dans les structures géologiques poreuses: étude numérique bidimensionnelle et stabilité des écoulements.* Thèse de l'Université de Bordeaux 1, 1989.

[11] Ya.G. Panovko, *Introduction to Theory of Mechanical Oscillations*, Nauka, Moscow, 1980 (Russian).

[12] I.G. Petrovskii, *Lectures on Theory of Ordinary Differential Equations*, Nauka, Moscow, 1970 (Russian).

[13] G.E. Shilov, *Mathematical Analysis. Second Special Course*, Nauka, Moscow, 1965 (Russian).

[14] J.H. Wilkinson, *The Algebraic Egenvalue Problem*, Clarendon Press, Oxford, 1965.

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